Simulation for fiber optics pulse propagation through a numerical solution of the Nonlinear Schrödinger Equation

Sierra Garzo, Èric and Mas Mendoza, Sergi
Grau en Enginyeria Física, ETSETB, Universitat Politècnica de Catalunya
(Dated: June 19, 2020)

Fiber pulse propagation is a common scenario in optical experiments. This article is a numerical simulation of such propagation using the split-step Fourier method with a brief discussion on its effects. To conclude, simulations with optical solitons are made since their special behaviour serve as a way to validate the numerical calculations.

I. INTRODUCTION

High speed and ultra-high capacity optical communications have recently emerged as essential in global information transmission networks. As the bit rate of the transmission system gets higher and higher, the modeling of proposed modulation techniques is very important so as to avoid costly practical demonstration [1]. That is when modeling pulse propagation becomes relevant.

Here we simulate the propagation of ultra short pulses thought a middle range fiber to feed an antenna to study how the initial pulse is altered through the fiber in order to neglect fiber effects to any experiment done with the antenna.

The pulse propagation in optical fibers is governed by three effects: attenuation, dispersion and non-linearity. These effects lead to the equation that will describe pulse propagation along the fiber, the Nonlinear Schrödinger Equation (NLSE). This partial differential equation has to be solved numerically to obtain a reliable simulation, commonly using the Split-Step Fourier method, treated later on.

II. THEORETICAL BASIS. THE NONLINEAR SCHRÖDINGER EQUATION

Like all electromagnetic phenomena, the propagation of optical fields in fibers is governed by Maxwell's equations. The wave equation can be obtained from these laws. After a series of simplifications, the propagating pulse is given by the solutions of the Helmholtz equation in frequency domain

$$\nabla^2 \tilde{E} + \beta(\omega)^2 \tilde{E} = 0, \tag{1}$$

where $\beta(\omega)$ is the wave number. This equation can be solved by using the method of separation of variables. So studying the equation one focuses on its propagation component $A(z,t)$, the affected component through the propagation, whose Fourier transform satisfies, after approximations like the Slowly-Varying-Envelope Approximation (SVEA),

$$\frac{\partial \tilde{A}}{\partial z} = i[\beta(\omega) + \Delta\beta_0 - \beta_0]\tilde{A}, \tag{2}$$

where $\Delta\beta_0$ term includes the effects of fiber loss and nonlinearity and $\beta_0 = \beta(\omega_0)$ is the wave number corresponding to the central frequency of the pulse. As the expression of $\beta(\omega)$ is rarely known, it is usual to develop $\beta(\omega)$ as its ω_0 centered Taylor series, in the form

$$\beta(\omega) = \beta_0 + \beta_1(\omega - \omega_0) + \frac{1}{2}\beta_2(\omega - \omega_0)^2 + \cdots \tag{3}$$

where

$$\beta_m = \left(\frac{d^m \beta(\omega)}{d\omega^m}\right)_{\omega=\omega_0} \qquad m = 0, 1, 2, 3, \ldots. \tag{4}$$

The cubic and higher-order terms in the expansion are negligible if the spectral width of the pulse satisfies the condition $\Delta\omega << \omega_0$. In general, they might be regions in which $\beta_2 \approx 0$ and the third order term should be taken into account.

After these simplifications, considering the fiber losses and nonlinearity parameters instead of $\Delta\beta_0$ and taking the inverse Fourier transform, equation (2) becomes

$$\frac{\partial A}{\partial z} + \beta_1 \frac{\partial A}{\partial t} + i\frac{\beta_2}{2}\frac{\partial^2 A}{\partial t^2} + \frac{\alpha}{2}A = i\gamma(\omega_0)|A|^2 A. \tag{5}$$

The attenuation constant α is the responsible of power fiber losses. It only plays an important role for long fibers, so it will be neglected in the calculations. The γ coefficient accounts for the nonlinear index coefficient. It is usually expressed in $W^{-1}km^{-1}$ and in our fiber (standard single mode fiber) takes the value $\gamma = 1.76W^{-1}km^{-1}$. The β_2 term is related to a well known parameter of optical fibers, the dispersion parameter (D), or GVD parameter, defined as $\frac{d\beta_1}{d\lambda}$. Is related to β_2 as

$$D = -\frac{2\pi c}{\lambda^2}\beta_2. \tag{6}$$

and its value is $D = 17ps/nm$ in our case.

The β_1 term is directly related to the group velocity v_g (in fact one is the inverse of the other) so changing the frame of reference to the one moving with the pulse by the time transformation $T = t - z/v_g$, the β_1 term disappears from the dispersion equation [2], leading to the final result (also, taking $\alpha = 0$), the Nonlinear Schrödinger Equation (NLSE)

$$\frac{\partial A}{\partial z} = -i\frac{\beta_2}{2}\frac{\partial^2 A}{\partial T^2} + i\gamma|A|^2 A \tag{7}$$

Equation (7) can be sumarized in

$$\frac{\partial A}{\partial z} = (\hat{D} + \hat{N})A, \tag{8}$$

where $\hat{D}$ is the time operator that accounts for dispersion in the linear propagation, called Group-Velocity Dispersion (GVD), and $\hat{N}$ is the time operator that governs the effect of fiber nonlinearities on pulse propagation, called Self-Phase Modulation (SPM) since the absolute value of the pulse itself modulate its propagation as in (7).

III. NUMERICAL SIMULATIONS. THE SPLIT-STEP FOURIER METHOD

Equation (8) is usually solved using the Split-Step Fourier method (SSFM). The main idea is to divide the pulse propagation in alternating dispersion and nonlinear steps, considering one at each time. If steps are small enough, the numerical solution will tend to the propagation of the pulse [3]. Before considering the general case, it is instructive to study the effects of dispersion and nonlinearity alone.

A. Dispersion Effects

The effects of GVD on optical pulses propagating in a linear dispersive medium are studied by setting $\gamma = 0$ in (7), an equation that is readily solved by using the Fourier-transformation

$$i\frac{\partial A}{\partial z} = \frac{\beta_2}{2}\frac{\partial^2 A}{\partial T^2} \iff i\frac{\partial \tilde{A}}{\partial z} = -\frac{\beta_2}{2}\omega^2 \tilde{A} \tag{9}$$

whose solution is

$$\tilde{A}(z,\omega) = \tilde{A}(0,\omega)e^{\frac{i}{2}\beta_2\omega^2 z} \tag{10}$$

where $\tilde{A}(0,\omega)$ is the Fourier transform of the initial pulse in $z = 0$. This makes dispersion really easy to work with in frequency domain. Therefore, a Fourier transform must be done to work with it.

Also, equation (10) shows that GVD changes the phase of each spectral component of the pulse by an amount that depends on both the frequency and the propagated distance. Even though such phase changes do not affect the pulse spectrum, they wide the pulse over its propagation. A clear exemplification of it is the Gaussian pulse. Considering the initial pulse shape as

$$A(0,T) = e^{-\frac{T^2}{2T_0^2}}, \tag{11}$$

with T_0 the half with (at $1/e$ intensity). Then, the amplitude at any point z along the fiber is given by

$$A(z,T) = \frac{T_0}{\sqrt{T_0^2 - i\beta_2 z}} e^{-\frac{T^2}{2T_0^2 - i\beta_2 z}}. \tag{12}$$

Therefore, a Gaussian pulse maintains its shape but widens as propagates and gets chirped, therefore higher frequencies will travel at different velocity than lower frequency, and it will be higher or lower depending on the sign of β_2.

B. Nonlinear Effects

The effects of SPM on optical pulses are studied by setting $\beta_2 = 0$ in (7). The equation is then

$$i\frac{\partial A}{\partial z} = -\gamma |A|^2 A. \tag{13}$$

This equation can be solved analytically considering $A(z,T) = V(z,T)exp(i\Phi(z,T))$ being $V(z,T)$ its modulus and $\Phi(z,T)$ its phase. After some transformations, the following two differential equations are obtained and are easy to solve

$$\frac{\partial V}{\partial z} = 0 \qquad \text{and} \qquad \frac{\partial \Phi}{\partial z} = \gamma V^2, \tag{14}$$

that give the solution

$$A(z,T) = A(0,T)e^{i\gamma|A(0,T)|^2 z}. \tag{15}$$

Equation (15) shows that while the pulse maintains its shape along its propagation in the time domain, SPM only changes the phase of the pulse by a quantity proportional to its squared absolute value and the propagated distance. However, the shape of the pulse in frequency domain will change as long as the pulse propagates through the fiber. This will incur in a phase shift that will generate a change in intensity. Therefore, nonlinear propagation widens the spectral shape of the pulse while maintains its intensity, as seen in Figure 1. That shows the duality between dispersion an nonlinear effects, seen also in equations (10) and (15), in the sense that while dispersion widens the pulse in time domain, nonlinear effects does it in frequency domain.

C. Split-Step Fourier Method

As said before, the Split-Step algorithm consists on propagating the pulse by steps of size h alternating dispersive regimes with nonlinear regimes by a half step difference. The basis of the propagation comes from equation (8). A formal solution of it is given by

$$A(jh,T) = e^{h(\hat{D}+\hat{N})}A((j-1)h,T). \tag{16}$$

As dispersive and nonlinear steps should be independently, the exponential should be separated in the product of the dispersive one and the nonlinear one. But this operators do not commute, so using the Baker-Hausdorff formula [4],

$$e^{h\hat{D}}e^{h\hat{N}} = e^{h(\hat{D}+\hat{N})+\frac{h^2}{2}[\hat{D},\hat{N}]+\frac{h^3}{12}[\hat{D}-\hat{N},[\hat{D},\hat{N}]]+\cdots}. \tag{17}$$

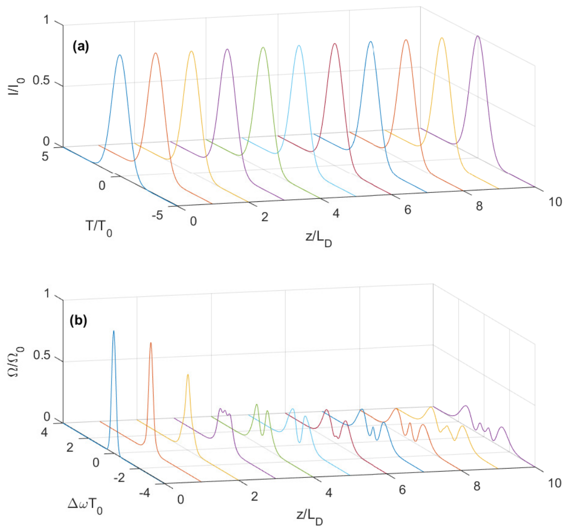

FIG. 1. Evolution of (a) pulse shapes and (b) optical spectra of an initially unchirped Gaussian pulse (C=0) considering no dispersion ($\beta_2 = 0$)

Therefore, if only the first term is taken into account, the separated split-Fourier formula is obtained. That is the reason why the method is accurate to the second order of h. The accuracy of the method can be improved considering propagating half a step the dispersion, then applying nonlinearity at half of the step and then again propagating half a step the dispersion. This is what is know as the symmetric Split-Step Fourier method. Considering this, the step propagation becomes

$$A(jh,T) = e^{\frac{h}{2}\hat{D}} e^{\int_{(j-1)h}^{jh} \hat{N}(z)dz} e^{\frac{h}{2}\hat{D}} A((j-1)h,T). \quad (18)$$

The integral of the nonlinear operator can be approximated as before by $h\hat{N}$ evaluated at the middle point with a negligible error if h is small enough. For the $h\hat{D}$, dispersion alone is applied for a half step $h/2$ both before and after the nonlinear operator. To do so, a Fourier transform with FFT is done to apply dispersion for a length of $h/2$ and then inverse transform to get the pulse in time domain again. In this case, the operators will apply straight forward, as exemplified in figure 2.

1. FFT algorithm and windowing

The Fast Fourier Transform (FFT) is a Fourier transform algorithm really fast under certain conditions. The income has to be a windowed zero-centered function discretized as a vector whose lenght is $N = 2^a$ for optimal efficiency, in this case using $N = 2^{11}$ points for the time domain. With the discretization of 2^a points, the algorithm becomes really efficient but it has an inconvenience. The fact that its a discret Fourier transform implies that,

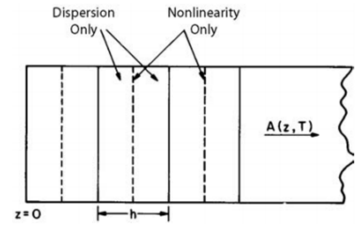

FIG. 2. Illustration of the Split-Step Fourier method. Fiber length is divided in length h segments. The effect of nonlinearity is included at the dashed lines. [4]

if windowing parameters are not good enough, aliasing might show up. Having N points with a temporary step Δx, the length of the temporary window is $L = N\Delta x$ and in the frequency domain gives

$$W = \frac{\pi}{\Delta x} \qquad \text{and} \qquad \Delta\omega = \frac{2\pi}{L} = \frac{2\pi}{N\Delta x}, \quad (19)$$

where W is the width of the frequency window and $\Delta\omega$ the frequency step [5]. The importance of this parameters is that, as seen, dispersion widens the temporal profile while nonlinear effects wide the optical spectrum, so both windows have to be wide enough for not having aliasing. But also both steps have to be small enough to have a good approximation of both temporal and spectral profiles, so temporal step has to be small enough but not too much because frequency step is inverse proportional to it. The temporal step used was $\Delta x = 0.05T_0$ but other parameters might be used in other situations to avoid aliasing.

D. Pulse fiber propagation

The effect of both dispersion and nonlinear effects becomes clear by the definition of a new parameter N, that resumes the effect of propagation on the pulse, such that

$$N^2 = \frac{L_D}{L_{NL}}, \quad (20)$$

where

$$L_D = \frac{T_0^2}{|\beta_2|} \qquad \text{and} \qquad L_{NL} = \frac{1}{\gamma P_0} \quad (21)$$

are the dispersion length and the nonlinear length respectively, that provide the length scales over which dispersive or nonlinear effects become important for pulse evolution. Figure 3 shows the pulse evolution for $N = 1$, when dispersion and nonlinear effects are similar. The effect produced is that time profile gets wider at the same time that spectral profile gets narrower. In the other hand, Figure 4 shows that when nonlinear effects are more important, the spectral profile gets wider. Therefore, although dispersion and nonlinear effects seem to separately affect different thing, when taken into account together they interact and produce different effects depending on the value of N.

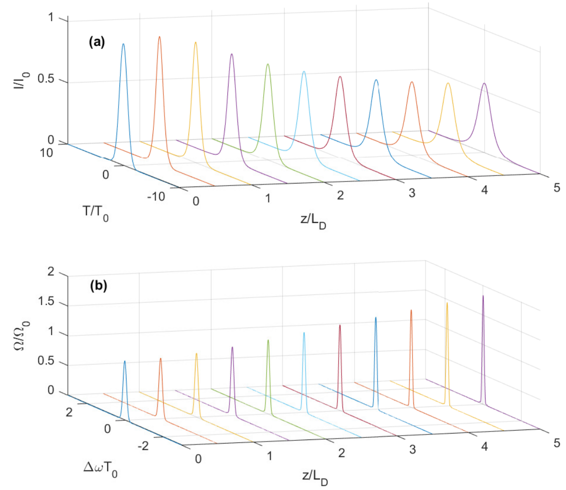

FIG. 3. Temporal (a) and spectral (b) evolution over a distance $5L_D$ for an initially unchirped Gaussian pulse with $\beta_2 < 0$ with parameters such that $N = 1$.

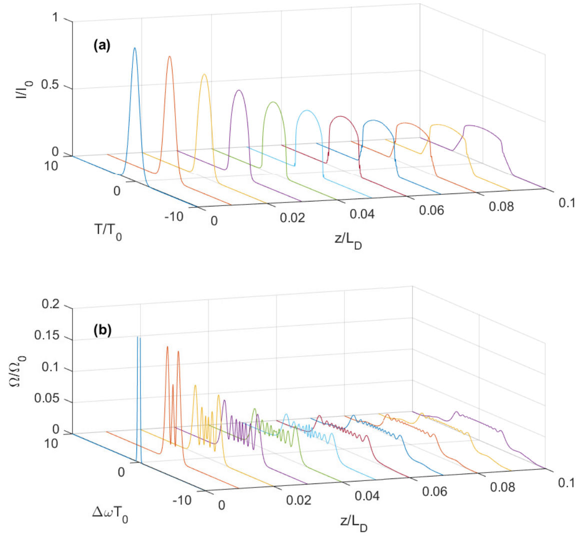

FIG. 4. Temporal (a) and spectral (b) evolution over a distance $0.1L_D$ for an initially unchirped Gaussian pulse with $\beta_2 > 0$ with parameters such that $N = 30$.

E. Special case: Solitons

It turns out that pulse fiber propagation has specific pulse-like solutions that either do not change along fiber length or follow a periodic evolution pattern. Such solutions are known as optical solitons [4]. After some calculations, it can be shown that the soliton shape is

$$A(0,T) = N \operatorname{sech}(T/T_0) \tag{22}$$

where N is defined in equation (20), and is also the soliton order (so an integer number) [4]. In figure 5 this behavior

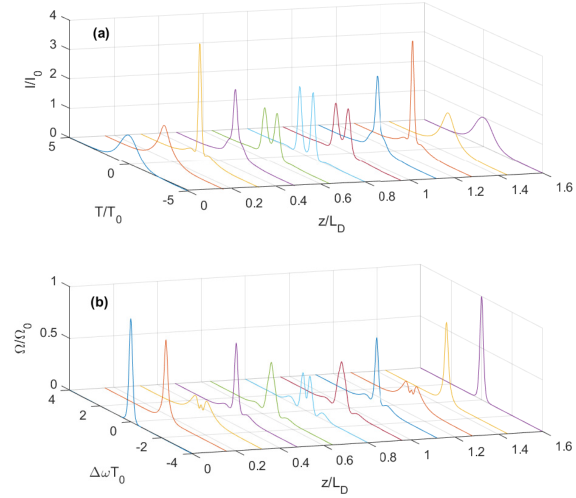

FIG. 5. Evolution of (a) pulse shapes and (b) optical spectra of a third-order soliton (N = 3) over one soliton period.

is observed. In this case, the pulse changes but follows a periodic evolution pattern in both temporal and spectral propagation. Note that the pulse split near $z/L_D = 0.5$ and its recovery in $z/L_D = 0.8$, the half of the soliton period.

IV. CONCLUSIONS

GVD and SPM are the two main effects of pulse propagation and both have concrete outcomes: GVD broadens the pulse while SPM does it with the spectral shape. This behavior has been observed in the simulations with Gaussian-shaped pulses for both cases alone, and then the same has been done including the two cases. Including the N parameter has simplified the way to understand these two effects and how they interact depending on its value. Finally, the special behavior of optical solitons and the fact that all simulations done perfectly agree with those present in [4] with the same parameters, have been used to validate the simulation.

The numerical simulation presented here could be improved by doing a more accurate calculation of the integral present in equation (18) by, for example, the usage of a trapezoidal approximation. This would need an iterative procedure as an evaluation of $\hat{N}$ in a spot that has not been calculated yet is needed.

V. ACKNOWLEDGMENTS

We would like to thank María Santos Blanco for her assessment and kind help throughout the work.

[1] L. BINH, International Journal of Communications, Network and System Sciences (SciRes) **2**, 97 (2009).

[2] Álvaro Fructuoso García, "Nonlinear pulse propagation in optical fibers," Universidad Politécnica de Cartagena and Norwegian University of Science and Tehcnology (2012).

[3] B. R. Washburn, *Dispersion and Nonlinearities associated with super continuum generation in microstructure fibers*, Ph.D. thesis, Georgia Tech (2002), chapter 4.

[4] G. P. Agrawal, *Nonlinear Fiber Optics*, 4th ed. (Academic Pressr, 2007).

[5] T. Kaiser, "An fft primer for physicists," Document of the Institue of Applied Physics, Abbe School of Photonics Friedrich-Schiller-Universitat Jena, Germany (2014).